Broker Routing Decisions in Limit Order Markets

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ABSTRACT

Many investors do not access equity markets directly; instead they rely on a broker who receives their order and submits it to a trading venue. Brokers face a conflict of interest when the commissions they receive from investors differ from the costs imposed by different trading venues. Investors want their orders to be filled with the highest probability, while brokers choose venues in order to maximize their own profits. I construct a model of limit order trading in which brokers serve as an agent for investors who wish to access equity markets. When routing liquidity taking orders (market orders), brokers preferentially route to venues with lower fees, driving up the execution probability at these venues and lowering adverse selection costs. However, when routing liquidity supplying orders (limit orders), if routing decisions are driven primarily by rebates from the exchanges, investors suffer from lower execution probability and higher costs of adverse selection. I find that when rebates for making liquidity are sufficiently similar across venues, routing decisions will be driven by the higher execution probability, while if rebates are sufficiently different, routing will be driven by rebates.

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Many investors, both retail clients as well as large institutional investors, do not access equity markets directly. Instead, they delegate the decision of which venue to trade on to their broker. In principle, the broker’s and client’s interest are aligned, in that the broker earns the commission only if the client’s order executes. However, it has been suggested that brokers may prefer to route a venue which maximizes their own profit, rather than the venue which best serves their client. Testifying before the United States Senate Committee on Homeland Security & Governmental Affairs, Robert Battalio raised the concern that brokers may be maximizing their intakes of rebates, paid to them by trading venues in exchange for order flow, rather than obtaining the best execution for their clients.\(^1\) In fact, concerns about order routing by brokers were raised as early as 2000 by the SEC,\(^2\) when they outlined proposed rule changes mandating disclosure of order routing practices.

The primary focus of this paper is to study the routing decisions that maximize brokers’ profits. Specifically, how these profit-driven routing decisions affect trader welfare and market quality. Trading fees are one distinguishing feature of trading venues for brokers. Orders either supply liquidity, by specifying a price and remaining available to other traders, or demand liquidity, by removing existing available orders. Recently, many venues have switched to “maker-taker” pricing frameworks, where traders are given a rebate if they supply liquidity, offset by a higher fee by those demanding liquidity. There also exist “inverse” or “taker-maker” exchanges, where traders demanding liquidity are provided a rebate, while those who supply pay a higher fee. Investors often pay their brokers a flat commission per trade, while the broker earns any rebates, or pays any trading fees charged by the venue. When clients delegate the choice of venue to their broker, a conflict of interest can arise from the presence of these trading fees.

To study the effects of broker routing decisions, I construct a two period model of limit

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order book trading, in which investors leave the routing decision to their broker and pay him a flat commission upon execution. Brokers have the choice of two possible venues for routing orders. The venues trade a single asset at fixed price levels and are each modelled based on the market in Foucault, Moinas, and Theissen (2007). Unlike the previous paper, these venues are differentiated by trading fees for making and taking liquidity. I assume that exchanges have the same net fee, defined as the taker fee plus the maker rebate. Thus, one exchange will have a lower taker fee, while the other will have a higher maker rebate.

In the first period, an uninformed investor arrives and maximizes her expected profit by choosing whether or not to submit a limit order to her broker. If the broker receives an order, he routes it to the venue which will maximize his expected profit. This is defined by the difference between the commission he charges the investor, and the trading fees charged by the exchange. In the second period, either an innovation in the asset value occurs and an informed trader arrives to remove mispriced limit orders from all markets or a liquidity trader arrives. If the liquidity trader arrives, he submits an order to his broker who again routes it to maximize his own expected profit.

In equilibrium, brokers route marketable orders to exchanges with lower taker fees, increasing the fill rate at these venues and lowering the risk of adverse selection for limit orders posted there. For limit orders, I find that unless the fees levied by the exchanges are very similar, brokers will route to the exchange with a higher maker rebate and lower execution probability for their clients. Intuitively, this follows from the broker’s profit maximization problem. When fees are similar, the broker benefits from the increased execution probability at the exchange with lesser rebates while incurring a low opportunity cost, as the rebate is only slightly higher at the alternative exchange. Conversely, when the fee structures are very different, the exchange with the higher rebate offers the broker sufficient profit upon filling the order to compensate for the lower execution probability.

Decision making by brokers impacts both their clients and trading venues as a whole. Preferential routing of uninformed market orders to exchanges with lower taker fees lowers
the adverse selection costs at these venues. Further, due to a lower concentration of informed trading, I find that the expected value of the trade, conditional on execution, improves for limit orders executed at these exchanges. In this case, the broker's decision on where to route uninformed market orders directly affects the market conditions for their limit order clients. Since informed traders are equally willing to trade at all exchanges, exchanges favoured by brokers for their uninformed market orders have improved fill rates.

I find that a number of factors improve for investors if fill rates, rather than rebates, drive brokers' limit order routing decisions. Specifically, more investors will choose to submit limit order, each order will face a lower expected adverse selection cost, and order execution will occur with a higher probability. Intuitively, this also follows from the improvement of market quality from an increase in the number of uninformed market orders. When brokers route their limit order investors based on fill rates, they route to the same exchanges where they route their uninformed market order clients. When market quality is improved, the expected value of a submitted limit order increases, making these attractive to a larger subset of possible investors. In environments where exchanges have very different fee structures, I show that by raising broker commissions, brokers will switch to routing based primarily on fill rates. When commissions rise, a broker's interests become increasingly aligned with those of his clients, as the profit from rebates becomes relatively smaller and the higher fill rate also brings an increased probability of earning the commission.

In an extended version of the model with multiple price levels and endogenous market makers, I confirm the results of the initial model. I find that bid-ask spreads at one exchange can be affected by increases or decreases in spreads at other exchanges. In addition, I find that limit order investor welfare increases when maker rebates are low, as market makers are no longer subsidized for providing liquidity.
A. Existing Literature

Existing work on maker-taker pricing can be divided into two groups. First is the work which focuses on the incentives for investors. Colliard and Foucault (2012) and Brolley and Malinova (2013) study maker-taker pricing regimes and their effects on investors. Colliard and Foucault (2012) construct a model of a frictionless market, and study the breakdown of the total exchange fee between maker and taker fees. They find that the only the total fee has an effect on investor outcomes, and that the breakdown between maker and taker fees has no impact on investor decisions or gains from trade. Brolley and Malinova (2013) construct an alternative model, where investors pay only a flat fee to a broker, who then passes the order on to a single exchange. They find that market orders sent to maker-taker exchanges are subsidized by investors who submit limit orders, when these investors pay a flat fee to their brokers. Empirically, exchanges with a maker-taker structure often have a better spread posted. (Malinova & Park, 2014; Anand, McCormick, & Serban, 2013) Exchanges with either a maker-taker structure or a higher taker fee have also been found to have a higher concentration of informed trading. (Yim & Brzezinski, 2012; Anand et al., 2013)

While the above work focusses on the incentives to investors who submit their orders directly to trading venues, the second group of work focusses on the outcomes when brokers select the venue. Empirical work by Boehmer, Jennings, and Wei (2007) focusses on the incentives to brokers created by the introduction of execution quality transparency requirements imposed by SEC Rule 11Ac1-5 (now Rule 605). They find that competition for order flow among broker-dealers drives orders to venues with high fill rates but also that many orders do continue to be routed to venues low execution quality.

The closest work to this paper is that of Battalio, Corwin, and Jennings (2014) who empirically study problem of the broker-client conflict of interest from trading fees. They find that brokers often make routing decisions based on the presence of liquidity rebates, rather than in the best interests of their clients. Further, they find that clients typically
face higher adverse selection costs at exchanges with higher liquidity rebates. The present paper provides theoretical confirmation of these two empirical results. First, I find that exchanges with higher liquidity rebates will endogenously have worse fill rates, and that a higher percentage of filled orders at the exchanges will be from informed traders. Second, I find that if trading fees are sufficiently different, brokers will route primarily based on rebates, rather than based on fill rates for their clients.

Significant regulatory attention has also been paid to trading fees and routing of investor orders by brokers. Since 2001, the SEC has required brokers to make details available regarding their routing practice through Rule 606. Further, the SEC requires brokers to disclose to investors, the routing details of their specific orders upon request, as well as statistics related to execution quality. Regulators have also taken an interest in Canada, where the Ontario Securities Commission (OSC) published proposed regulatory changes, which included a pilot study on prohibitions of maker-taker pricing structures, and disclosure of broker routing practices.

The remainder of the paper proceeds as follows. In Section 1, I describe a model with a single price level at the bid and the ask and exogenous market making. In Section 2, I describe the equilibrium results of this model. In Section 3, I present comparative static results. In Section 4, I construct an extended model with multiple price levels and endogenous market makers and in Section 5 I conclude.

I. Model

The model borrows components from the fixed-tick model in Foucault et al. (2007), as well as from trading fee models in Colliard and Foucault (2012) and Brolley and Malinova (2013).

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3Originally SEC Rule 11Ac1-6
A. Asset

There is a single asset in the model, which starts $t = 1$ with a value of $v$ and ends $t = 2$ with a value of $V$. With probability $\delta$ an innovation in the asset value occurs, raising or lowering the value of the asset with equal probability by amount $\sigma$ while with probability $1 - \delta$ no change occurs.

The asset trades on two exchanges at fixed ticks of size $\Delta$. The prices on each exchange are identical, with one price at the ask ($v + \Delta$), and one price at the bid ($v - \Delta$). There is room for a single order of quantity $Q_1 = 1$ (a buy limit order) or $Q_1 = -1$ (a sell limit order) at each tick on each exchange. Each exchange charges fees $M_i$ and $T_i$ to traders for orders making liquidity (limit orders) and taking liquidity (market orders) respectively. Both exchanges have the same total cost per order: $M_i + T_i = e \forall i$.

The innovation exceeds the tick size, such that all limit orders at the bid are mispriced if $V = v - \sigma$, and all limit orders at the ask are mispriced if $V = v + \sigma$. The risk that an informed trader picks off a mispriced order gives rise to adverse selection within the model.

B. Market Participants and Timing

The first period, is the liquidity supply period, in which agents post limit orders. The second period is the liquidity demand period, in which agents submit market orders. The timing of the model is illustrated in Figure 1.

A. Limit Order Investors. A utility-maximizing limit order investor arrives at the market at the beginning of period $t = 1$. She is unable to interact directly with the market and must instead post her order through a broker. Investors are risk neutral, do not discount, and gain utility only in relation to the asset being traded.

Analogously to Parlour (1998), limit order investors arrive with a desire to buy or sell with equal probabilities in the form of a quantity signal $Q_1 \in \{-1, 1\}$. As in Parlour (1998),
This figure illustrates the timing of this model. Investors and market makers provide liquidity in $t = 1$, while liquidity demand takes place in $t = 2$. With probability $\delta$ the liquidity demander is an informed trader, while with probability $1 - \delta$ he is uninformed.

Foucault (1999), she also receives a private value $y \in Y$, where $Y$ is a uniform distribution centred on zero.

She may submit a limit order for quantity $Q_1$, at a given price $P$, which will remain in the book until the end of $t = 2$. Orders at $Q_1 = 1$ are at the price $v - \Delta$ while orders of quantity $Q_1 = -1$ are at the price $v + \Delta$. She pays the broker a commission $c$, upon successful execution of an order.

When submitting an order to a broker, she considers two strategic factors. (1) To which exchange the broker will prefer to send market orders. (2) To which exchange the broker will route her limit order. By considering market order routing, she determines $\theta_i$, the probability her order will be executed if routed to exchange $i$.

A limit order investor who submits a limit buy order has expected utility:

$$U_{LO} = \theta_i \left( E[V|Ex_i] + y - (v - \Delta) - c \right)$$

where $E[V|Ex_i]$ is the expected value of the asset, conditional on the broker routing the order to exchange $i$ and subsequent execution in $t = 2$.

B. Brokers. Two risk neutral, uninformed, profit maximizing brokers exist in order to provide market access for investors. One broker routes limit orders during $t = 1$, while a second routes market orders during $t = 2$. The brokers receive orders from investors, and
route them to one of the trading venues.

When routing orders, the two brokers must follow two rules: (1) Brokers must accept and place all orders; and, (2) brokers may route limit orders to any venue, at the price specified by the order.

Brokers incur all costs from the exchanges to which they route orders to ($M_i$ for limit orders and $T_i$ for market orders at exchange $i$). In turn, they profit from the difference between these costs and the commission which they charge clients per order executed, $c$, which is given exogenously. For orders routed to exchange $i$, the broker’s profits for market order and limit orders are given by:

$$\pi_{MO} = [c - T_i] \quad \pi_{LO} = \theta_i[c - M_i]$$

C. Market Makers. An uninformed market maker arrives at the end of $t = 1$ in order to provide liquidity at both markets. The market maker exogenously submits limit orders at all empty price levels at both market. (This assumption is later relaxed in Section 5.) This market maker has direct access to the markets and pays fees associated with making liquidity ($M_i$).

D. Informed Traders. If an innovation occurs (probability $\delta$), an informed traders arrives at the market and views the current innovation. This trader has direct access to the market, uses market orders, and pays the fees associated with taking liquidity ($T_i$). This trader presents an adverse selection risk to limit order traders who have already placed orders, similar to Glosten and Milgrom (1985), Easley and O’Hara (1987), Glosten (1994) and others.

If the innovation is positive, such that $V = v + \sigma$, the trader immediately submits market orders for the mispriced orders at both markets at price $v + \Delta$. If the innovation is negative, such that $V = v - \sigma$, all orders at price $v - \Delta$ are filled. The informed trader then immediately leaves the market.
E. Liquidity Traders. If no innovation occurs, (probability $1 - \delta$) a liquidity demanding investor arrives with a quantity signal, distributed evenly over $Q_2 \in \{-2, -1, 1, 2\}$. He immediately submits market orders for the total amount of his desired quantity. This order is then routed by the broker, the trades execute, and he leaves.

II. Equilibrium

In this model an equilibrium consists of: (1) A solution to the broker’s profit maximization problem, and; (2) a solution to the limit order investor’s utility maximization problem. The solution to the broker’s problem consists of decisions on where to route market and limit orders from clients. The solution to the limit order investor’s problem consists of a decision as to whether or not submit an order, for every quantity $Q_1$, private signal $y$, and broker routing strategy. I make four simplifying assumptions on the parameter set.

**Assumption 1:** The total cost per order is set to $e = 0$.

**Assumption 2:** Without loss of generality, exchange 2 charges a higher taker fee, $T_1 < T_2$.

**Assumption 3:** The prices and fees are such that, if an innovation occurs, the value of the asset falls outside the grid of prices and fees $v < v + \Delta + T_i < v + \sigma$.

**Assumption 4:** The broker commission is such that $c > |T_i|, |M_i|$.

Assumption 1 implies that the exchanges behave competitively and have zero marginal cost of processing a trade. Relaxing this assumption creates a spread between the maker and taker fees at all exchanges, but does not significantly impact the results or interpretation of the model. Assumption 2 simplifies the solution for market orders, and implies that the broker will preferentially route market orders to exchange 1. Assumption 3 ensures that all orders are profitable for the informed trader. This assumption removes the case where taker fees are sufficiently high at one market that it is not affected by adverse selection. Assumption 4 ensures that all orders are profitable for the broker and ensures broker participation. Assumption 4 is not without loss of generality as its relaxation allows for the cases where brokers earn negative profit in expectation and can create situations in
which brokers prefer a low fill rate in order to minimize losses.

Theorem 1 (Existence of an Equilibrium):

(1) For fixed parameters $M_2, \delta, \sigma, c$, there exists a unique threshold maker fee $\overline{M}_1$, such that if $M_1 \leq \overline{M}_1$, brokers will optimally route limit orders to exchange 1.

(2) For fixed parameters $M_2, \delta, \sigma, c$, there exists a unique threshold private value $\overline{y}_i$, for each exchange $i$, such that for all $y \geq \overline{y}_i$ limit order investors with $Q_1 = 1$ will choose to submit a limit buy order.

Proof. The equilibrium is obtained through two steps in backwards induction. (1) The routing of market orders determines the broker’s expected profit for limit orders at exchanges 1 and 2. This determines the threshold $\overline{M}_1$. (2) Given the threshold $\overline{M}_1$, the limit order investor anticipates that her limit order will be routed to exchange $i$ and determines the expected utility of order submission. This determines the threshold $\overline{y}_i$. The full proof is in Appendix 1. \qed

A. Broker’s Routing Decisions

In order to maximize profits, the broker attempts to maximize the expected difference between the cost to clients ($c$) and the fees paid to exchanges ($M_i, T_i$).

Step 1: Market Order Routing. Brokers will route market orders in order to maximize:

$$\pi_{MO} = c - T_i$$

As there is only a single order available at any given tick, brokers must split up market orders of size $|Q_2| > 1$ across multiple venues.

Market order routing has direct consequences for limit orders. All else equal, brokers will route to exchanges with lower liquidity taking fees. This decision increases execution probabilities for limit orders at these exchanges. Further, both exchanges receive the same
This figure illustrates the equilibrium actions and pay-offs for limit order buyers, and their brokers. The equilibrium is determined through backwards induction. Given the expected pay-offs, the broker will choose to route limit orders to exchange $i$. Given the broker’s routing decision, the limit order investor will submit an order if her private value is above $y_i$. Actions and pay-offs for limit order sellers are symmetric.

absolute quantity of informed orders, but different quantities of uninformed orders. As a result, the expected value for the asset, conditional on execution, will be different across the two exchanges.

Proposition 1 (Market Order Routing and Limit Order Execution Probability):

The execution probability for limit orders at exchange 1, the low taker fee exchange, is always higher than on exchange 2 $\theta_1 > \theta_2$. The broker’s profit, conditional on execution, will be higher at exchange 2.

Proposition 1 results directly from the preferential routing of market orders to the venue with the lower taker fee. Since only large orders will be sent to both markets, limit orders posted at market 1 necessarily have a higher fill rate. The difference in quantity of market orders sent to each in exchange provides the driving force behind the principal-agent problem between brokers and their clients. When routing limit orders, brokers will always have the choice between one exchange with a higher execution probability for their clients (market 1) and an exchange with a lower fee for them (market 2).
It could be argued that, practically speaking, marketable orders from retail traders comprise only a small portion of total order flow. However, retail order flow is especially important in the context of adverse selection risk, as it generally contains less information content. As such, any exchange which receives a higher volume of retail orders compared to other orders likely has lower information costs for limit orders posted there.

**Step 2: Limit Order Routing.** When routing limit orders, brokers choose exchange $i$ in order to maximize:

$$\pi_{LO} = \theta_i [c - M_i]$$  \hspace{1cm} (4)

For limit orders, brokers will weigh the trade-off between a higher execution probability and a lower making fee. This trade-off, should it arise, is at the heart of the broker-client conflict of interest. Unlike market orders, brokers are able to route limit order to venues with a low probability of execution. If one exchange has maker fees which are sufficiently low compared to others, brokers may route there even if the execution probability is low.

Proposition 2 (Market Order Routing and Expected Asset Value):

*For limit buy orders, the expected value of the asset conditional on execution, is higher if it is routed to exchange 1, than if it is routed to exchange 2 ($E[V|Ex_1] > E[V|Ex_2]$). The result is reverse for limit order sells.*

Proposition 2 results from the probability of informed trading is higher at exchanges with maker fees that are more attractive for the broker. Since informed traders are willing to remove mis-priced orders from all exchanges, regardless of fee structure, exchanges with a lower number of retail orders face relatively higher adverse selection costs. This result supports the conclusions of empirical work from Anand et al. (2013) and Battalio et al. (2014), who find that exchanges with a maker-taker structure have a greater concentration of informed order flow.

Proposition 2 expands on Proposition 1 in relation to the effect of routing on brokers clients. Not only do investors suffer a lower execution probability if routing is driven by
rebates, but a larger portion of those executions result in adverse selection. Therefore, the expected value of the trade, conditional on execution, is lower if routing is driven by rebates.

B. Limit Order Investor’s Problem

A limit order investor views her quantity signal $Q_1$, her private value $y$, the broker price $c$ and anticipate the brokers’ routing strategies. Using these, she decides whether or not to submit a limit order:

$$\theta_i \left( E[V|Ex_i] + y - (v - \Delta) - c \right) > 0 \quad (5)$$

As described in Theorem 1, there exists a value $\overline{y}_i$, such that any limit order buyer with a sufficiently large private value $y$ will optimally choose to submit a limit order, given that it will be routed to exchange $i$.

Proposition 3 (Probability of Limit Order Submission):

*Investors require less favourable private valuations to submit an order if their order will be routed to exchange 1 ($\overline{y}_1 < \overline{y}_2$).*

Proposition 3 describes the difference in decision making behaviour of limit order investors based on broker routing. Limit order traders will optimally choose to submit an order if their expected utility, given routing and execution, is positive. As both the probability of execution and expected value given execution are higher if routed to exchange 1, more traders are willing to submit if their order will be routed there.

The decease in trader volume when routing for rebates also suggests a dilemma for brokers. If they were able to commit to routing to exchange 1, despite a lower profit per order, a larger number of clients would choose to submit orders. Depending on the parametrization, this can lead to a larger expected profit for brokers and the preferred routing scheme for their clients. The increase in the number of clients when orders are routed to exchanges with higher fill rates suggests an important role for routing disclosure by brokers, which may
ultimately serve as a commitment mechanism to their clients.\textsuperscript{5}

By combining with Propositions 2 and 3, a statement about welfare can be made.

Proposition 4 (Limit Order Investor Welfare):

*If the two markets are sufficiently similar ($M_1 \leq M_1$), more limit orders will be submitted by investors and each investor will be better off in expectation than if the two markets are sufficiently different ($M_1 > M_1$). That is to say, that the expected welfare of limit order investors is higher when markets are similar.*

For simplicity, I define welfare in a strictly utilitarian sense, insofar as total investor welfare is simply the sum of utility of each individual investor. However, in Proposition 4, not only does total welfare increase, but every individual investor also has higher expected utility. The increase in utility corresponds to a first-order stochastic dominance relationship, in which the distribution of investor utilities with two similar markets, first order stochastic dominates the distribution of utilities under two very different markets.

III. Comparative Statics

Of particular interest within this model is the adverse selection. An increase in $\delta$ reflects an increase in the probability that an information event occurs, and thus an increase in an uninformed trader’s risk of being adversely selected.

Proposition 5 (Adverse Selection and Limit Order Investors):

1. *When adverse selection rises limit order investors receive lower expected values for their trades, fewer investors are willing to submit orders, and the total welfare of investors declines.*
2. *When adverse selection rises, exchanges must be increasingly similar for brokers to route based on fill rates rather than rebates.*

Proposition 5 demonstrates the two negative effects of adverse selection on limit order

\textsuperscript{5}An example is SEC Rule 606, which mandates the partial disclosure of routing information for non-directed orders by brokers. Information related to this is available online at: http://www.ecfr.gov/cgi-bin/text-idx?SID=6d079725b329f4fc5ad9c80affb45d9f&node=se17.4.242_1606&rgn=div8
investors. First, as the probability of trading against an informed trader increases, the expected value of the trade is worse for limit order investors. The lower expected value lowers the number of limit order investors willing to submit orders to their brokers.

Secondly, during times when more information is reaching the markets, such as during periods of announcements, brokers are more likely to route to based on rebates. The increased probability of informed trading levels the total probability of execution across all exchanges and causes brokers to concentrate their orders at exchanges with high maker rebates. These exchanges, in turn, have the highest adverse selection risk to investors, increasing the impact to clients. That is to say, that brokers increase their clients’ risk of being adversely selected during periods when adverse selection is highest.

Proposition 6 (Increase in Broker Commission):

There exists a threshold \( c \), such that for all \( c > c \) brokers route limit orders to exchange 1.

Proposition 6 reflects one of the basic ideas in principal-agents problems. Given a sufficient incentive, here in terms of a higher commission, the brokers’ interests will be aligned with their clients interest. In conjunction with Proposition 2, this implies that, in some cases a higher commission may increase welfare for both brokers and their clients. Though clients generally prefer paying a lower commission, it implies that when the commission is too low, they may suffer from a conflict of interest.

Proposition 7 (Broker Commissions and Welfare):

Consider a market in which brokers charge their clients \( c < c \). If the following condition holds:

\[
    c - c \leq [V|Ex_1] - E[V|Ex_2]
\]

(6)

an increase in the commission from \( c \) to \( c \) increases the expected profit of the broker and the expected utility of all investors.

Proposition 7 describes when brokers and their clients can be made better off by an increase in commissions. The increase in commission causes the broker to begin routing to exchange
which lowers the adverse selection costs to investors. If Equation 6 is satisfied, the increase in commission is totally offset by the decrease in adverse selection costs.

There is a second interpretation to the broker commission, which stems from the costs incurred by the broker. Aside from trading fees, the costs to brokers of processing an order are set to zero in this model. In a situation where the broker is able to reduce his costs, either through more efficient internal behaviour or through lower regulatory or other costs, this is effectively equal to an increase in the commission he receives per order, without an increase in the commission each client pays. In this case, a decrease in non-trading fee costs to the broker increases his profit from the commission upon execution, and may also cause him to optimally route to the exchange with a higher fill rate.

IV. Extension: Endogenous Market Making

In the benchmark model, the price grid at each side of the market was of size one. In this extension, the prices on each exchange are identical, and I allow for two ticks at the ask \((v + \Delta, v + 2\Delta)\), and two at the bid \((v - \Delta, v - 2\Delta)\). There is room for a single order of quantity \(Q_1 = 1\) (a buy limit order) or \(Q_1 = -1\) (a sell limit order) at each tick on each exchange. Further, I allow market makers to choose whether or not to post at each available price level.

A. Market Participants

A. Market Makers. Uninformed market makers provide liquidity in each market. The market makers arrive immediately after the investor’s order is routed in \(t = 1\), and are able to place further orders. Market makers do not go through the broker and instead, pay fees directly to the exchange.

Market makers view the order, if any, placed by the limit order investor, and immediately place orders at any tick which, given the expected behaviour of all other agents, gives a positive expected value. For buy limit orders, this is any price \(P\) such that:
\[ E[V|E_{x_i}] \geq P - M_i \]  \hspace{1cm} (7)

**B. Limit Order Investors.** The limit order investor may now choose to submit an order at price \( P \) in order to maximize her utility:

\[ U_{LO} = \theta_i(P) \left( E[V|E_{x_i}] + y - P - c \right) \]  \hspace{1cm} (8)

For buyers, price \( P \) is either \( v - \Delta \) or \( v - 2\Delta \). As before she may also choose to abstain from trading.

When submitting an order to a broker, she now considers three factors. (1) To which exchange the broker will route her limit order; (2) To which exchange the broker will send market orders; (3) Which price levels market makers will post at. Combining these three, she determines \( \theta_i(P) \), the probability that her order will execute at a given price \( P \) if routed to exchange \( i \).

**B. Equilibrium**

In the extended model an equilibrium consists of: (1) A solution to the broker’s profit maximization problem, and; (2) A solution to the limit order investor’s utility maximization problem; (3) A decision by market makers to post, or not, at every empty tick.

**Assumption 5:** The grid of prices is such that \( \Delta + T_i < \sigma < 2\Delta + T_i \) and \( \sigma < 2\Delta - M_i \).

Assumption 5 ensures that orders placed at the farther tick are not adversely selected against. Informed traders will only pick off orders at the closest tick and market makers will always be willing to post at the farther ticks.

**Theorem 2 (Equilibrium of the Full Model):**

1. For fixed parameters \( M_1, M_2, \delta, \sigma, c \) there exists a unique market making plan at exchange 1 \( \tilde{M}_1 \), such that market makers will choose to post at prices \( \pm \Delta \) at exchange 1 if \( M_1 \leq \tilde{M}_1 \).

For fixed parameters \( M_1, M_2, \delta, \sigma, c \) and market making plan \( \tilde{M}_1 \), there exists a market making
plan at exchange 2 $\tilde{M}_2$, such that market makers will choose to post at prices $\pm \Delta$ at exchange 2 if $M_2 \leq \tilde{M}_2$.

(2) For each market making plan and fixed parameters $M_2, \delta, \sigma, c$, there exist unique threshold maker fees $\overline{M}_1(\Delta), \overline{M}_1(2\Delta)$ such that if $M_1(P) \leq \overline{M}_1(P)$ brokers will optimally route limit orders at prices $P$ to exchange 1. Otherwise brokers will route limit orders at price $P$ to exchange 2.

(3) For each market making plan and fixed parameters $M_2, \delta, \sigma, c$, there exist unique threshold private values $\overline{y}_i(\Delta), \overline{y}_i(2\Delta)$, for each exchange $i$, such that investors with $Q_1 = 1$, will submit an order at price $-\Delta$ if $y \geq \overline{y}_i(\Delta)$ and at price $-2\Delta$ if $\overline{y}_i(\Delta) > y \geq \overline{y}_i(2\Delta)$. Otherwise, if $y < \overline{y}_i(2\Delta)$, limit order buyers will abstain.

A. Market Makers. Given the expected routing of market orders, market makers choose whether or not to post at each empty tick in order to solve their profit maximization problem. Since there are many market makers, they individually choose whether each tick they may post at is profitable. Market maker behaviour depends, in particular, on the difference in fees between the two markets. Market makers are aware that brokers will preferentially route market orders to the exchange with lower taker fees, lowering the chance of being adversely selected at these exchanges.

In equilibrium, there are four possible cases for market making, which will be referred to throughout the remainder of the section. Market makers are always willing to post at the far ticks at both exchanges, and therefore the cases are defined by their willingness to post at the narrow ticks. (1) Market makers post at all ticks at both markets; (2) Market makers only post at the narrow tick of the high maker rebate exchange (3) Market makers only post at the narrow tick of the high fill rate exchange; (4) Market makers only post at the far ticks.

These market making cases are defined by fee thresholds $\tilde{M}_1, \tilde{M}_2$. The fee thresholds exist such that, if $M_i < \tilde{M}_i$, market makers post at the narrow ticks at exchange $i$.
This figure represents the possible equilibria for market makers. Case 1: Market makers are willing to post at the narrowest ticks in both markets, $\Delta$. Case 2: Market makers are willing to post at the narrow tick at market 2, $\Delta$, but only at the farther $2\Delta$ in market 1. Case 3: Market makers are willing to post at the narrow tick at market 1, $\Delta$, but only at the farther $2\Delta$ in market 2. Case 4: Market makers are only willing to post at the farthest ticks in both markets, $2\Delta$.

(1) If $M_1$ increases from $M_1 \leq \tilde{M}_1$ to $M'_1 > \tilde{M}_1$: (i) Market makers will no longer post at prices $\pm \Delta$ at exchange 1; (ii) $\tilde{M}_2$ rises and market makers will begin to post at prices $\pm \Delta$ at exchange 2, if they had not already been doing so.

(2) If $M_2$ increases from $M_2 \leq \tilde{M}_2$ to $M'_2 > \tilde{M}_2$ market makers will no longer post at prices $\pm \Delta$ at exchange 2.

Proposition 8 describes the results shown in Figure 3. Exchange 1 receives a larger proportion of uninformed market orders, and the market makers’ decision to post at this exchange affects the execution probability at exchange 2. This is best seen on the transition from Case 3 to Case 2 in Figure 3. The rise in the fee at exchange 1 to $M_1 > \tilde{M}_1$ causes the market makers to cease posting orders at the narrow price levels at exchange 1. This increases the number of uninformed orders that reach exchange 2, and decreases $\tilde{M}_2$ such that market makers will optimally post at the narrow price levels.
These results are driven by the principle of order protection for market orders and provide policy insight. Specifically, the change in fee structure at one exchange may influence the spreads at the competing exchanges. In the case presented in Figure 3, exchange 1 becomes a taker-maker exchange and market makers are no longer willing to post at the best. Order protection moves market orders to exchange 2, because it quotes a narrower spread.

**B. Execution Probabilities.** If the trader chooses to submit at the narrowest tick, there is a risk that when the asset value undergoes an innovation, the order will be picked off. This occurs with probability $\frac{1}{2} \delta$. Given the optimal market order routing strategy, the probability $\theta_i(P)$ is a function of the ticks that market makers are willing to post at. Table I gives the execution probabilities for limit orders, at each ask price, given market maker behaviour.

**C. Limit Order Investors.** As in the initial model, limit order traders consider the behaviour of all other agents in the model when deciding whether to submit an order. The presence of endogenous market makers complicates their decision, as their presence affects the execution probability of their limit order, seen in Table I.
This figure represents the expected utility of the limit order investor, under varying exchange fees. Investors have higher expected utility values when exchanges are similar and when exchange fees force market makers to post wider spreads.

Figure 4 illustrates two separate effects. The first effect occurs when the trading fees are similar at both venues. This region corresponds to where $M_1(\Delta) \leq \overline{M}_1(\Delta)$ and $M_1(2\Delta) \leq \overline{M}_1(2\Delta)$. In this region, brokers optimally route orders to the exchange with the higher fill rate, rather than the higher maker rebate (or lower maker fee), and investor welfare is higher. The second effect occurs on the left and lower borders, where maker fees are high. In these regions investors are better off since market makers no longer post at the narrowest ticks at one, or both venues.

Proposition 9 (Limit Order Investor Welfare in the Extended Model):

(1) If $M_1$ increases from $\overline{M}_1 < M_1 \leq \bar{M}_1$ to $M'_1 > \bar{M}_1$, then limit order investor utility increases.

(2) If $M_2$ increases from $M_2 \leq \bar{M}_2$ to $M'_2 > \bar{M}_2$, then limit order investor utility increases.

The increase in investor utility described in Proposition 9 comes from two sources. First, when $M_1$ increases such that market makers no longer post at the narrow ticks at exchange
1, the proportion of uninformed orders increases at exchange 2. There is an increase in both the expected value of limit orders routed to exchange 2 and the proportion of investors willing to submit an order. Second, when either $M_1$ or $M_2$ increases, such that market makers no longer post at the narrow ticks of one exchange, liquidity declines at the best. Some investors who were previously unwilling to submit orders will then choose to submit orders at the wider price levels ($\pm 2\Delta$), increasing welfare.

Rebates allow market makers to post more aggressively, and in turn give them a competitive advantage against limit order investors, who don’t receive rebates. It is important to note, that narrower spreads are advantageous to many market participants, notably those trading with market orders. Therefore, while limit order investor utility does increase when market makers choose not to compete at tight spreads, the outcomes for other investors whose utility is not addressed by this model may decline.

V. Conclusion

The principal-agent relationship between brokers and their clients is one which has the potential to impact both the individuals involved and markets as a whole. While existing theoretical literature has addressed the concept of exchange fees, specifically maker-taker pricing, a gap remains in explaining how these fees and rebates drive broker behaviour and affect their clients. In order to explain these impacts I construct a static model of limit order trading in which brokers route limit orders from their clients to one of two venues.

I show that in an environment with fixed price levels, rebates for making and taking liquidity are able to drive broker routing decisions for both limit and market orders. These routing decisions, in turn, effect both the fill rates and relative probability of informed trading at both exchanges. Fill rates are higher for limit orders placed at exchanges with smaller maker rebates while the probability of facing informed orders is higher at exchanges with larger maker rebates.

I find that when exchanges have similar fee structures, brokers have less incentive to
deviate from their clients interests, and will optimally route to the exchange with a higher fill rate. In this case, more of their clients will submit orders, each order will have a higher expected value for the client, and investor welfare will be higher. On the contrary, when exchanges have sufficiently different fee structures, routing will be driven by liquidity rebates, and investor welfare will be lower.

The decision to route is also influenced by the broker’s commission, given exogenously in this model. The results show that when commissions are higher, brokers interests become aligned with their clients, as they profit from a higher fill rate.

These results are furthered in an extended model with multiple price levels and exogenous market making. In this environment limit order investors also benefit when both exchanges charge high liquidity making fees. When maker fees are high, market makers are no longer subsidized when making liquidity, and are less willing to provide liquidity at narrow spreads. Consequently, limit order investors, who pay only flat fees to their brokers, face less competition for execution of their orders and have a welfare improvement.

Also in the extended model, I find that changes in the fee structure at one exchange, may influence the spread at a second exchange. This occurs through the shifting of uninformed traders across exchanges. The utility effects to limit order investors from this change depend on the direction of the shift.

This model has a number of implications for policy regarding both trading venues and brokers. First, it implies that the proliferation of trading venues may not necessarily be beneficial for investor welfare, and in fact may be harmful in the case where certain venues face higher adverse selection costs or lower fill rates. Second, while brokers may be restrained in some of their decisions (such as by the Order Protection Rule), it is important to take factors other than price into account when defining concepts such as best execution. Specifically, it is beneficial to investors to consider factors such as fill rates when brokers select venues for their clients. Third, the change in the fee structure at any one exchange, can influence the spreads and market conditions at other exchanges, and that these changes should not and
do not occur in a vacuum.

It is important to caution against the implication that fewer trading venues will unambiguously increase investor welfare. One of the primary assumptions in the model, is that all exchanges have the same total fee per order and that this fee is set exogenously. In reality, different venues have different spreads between their maker and taker fees, which result from the competitive environment between them. In the extreme case of a single monopoly exchange, this spread would likely increase, and the additional costs would likely be passed on by brokers to their clients. Further, there are other features of trading venues, which I do not model, that may appeal to either brokers or their clients.\textsuperscript{6}

\textsuperscript{6}Examples include venues with decreased latency, dark venues, venues with block trading or venues which operate on an auction framework.
References


VI. Appendix 1: Proofs

A. Primary Equilibrium

Lemma 1 (Market Order Routing Equilibrium):

In equilibrium, brokers will treat each unit of a multiple unit market order as a separate, order of size one. Brokers will route market orders in the following manner.

- For orders of size $|Q_2| = 1$, to whichever exchange has the lowest liquidity taker fee, which will be exchange $i = 1$ since $T_1 \leq T_2$.

- For orders of size $|Q_2| = 2$, to both exchanges.

Lemma 2 (Limit Order Routing Equilibrium):

For limit orders, brokers are able to route, at the price dictated by their client ($P$), to any exchange. In equilibrium, brokers will route limit orders in the following manner.

1. In order to maximize profit, brokers will route an order to whichever exchange has the highest expected profit:

   \[
   \theta_i[c - M_i] \geq \theta_j[c - M_j]
   \]  

\textit{Proof.} Investors are unable to contract with brokers and brokers choose routing behaviour following receipt of an order. Suppose a broker were to agree to route to exchange 1 where $\theta_1 > \theta_2$ but $\theta_1[c - M_1] < \theta_2[c - M_2]$:

- If $\bar{y}_1 < \bar{y}_2$ it is possible that more orders would be submitted and total expected profit over all possible investors would be higher such that $Pr(y \geq \bar{y}_1)\theta_1[c - M_1] > Pr(y \geq \bar{y}_2)\theta_2[c - M_2]$.

- However, for any given order $\theta_1[c - M_1] < \theta_2[c - M_2]$ and the broker has incentive to deviate following the receipt of the order.
Therefore, the broker’s promise to route to exchange 1 is not credible, unless every order sent there is more profitable.

Lemma 3 (Equilibrium Limit Order Trading Decisions):

*If a limit order investor arrives as a buyer, she will submit an order if her expected utility, conditional on routing and execution, is positive. For an order being routed to exchange $i$, this occurs if:*

$$\theta_i (E[V|Ex_i] + y - (v - \Delta) - c) > 0$$  \hspace{1cm} (10)

**Proof of Theorem 1 Part 1**

By Lemma 2, the broker will optimally route to exchange 1 iff:

$$\theta_1[c - M_1] \geq \theta_2[c - M_2]$$ \hspace{1cm} (11)

By substitution of $\theta_i$ and algebraic manipulation this gives the condition that:

$$M_1 \leq c \frac{1 - \delta}{2} + M_2 \frac{1 + \delta}{2}$$ \hspace{1cm} (12)

Denoting $M_1 = \overline{M}_1$ when this condition holds with equality, the broker will optimally route any limit order to exchange 1 if $M_1 \leq \overline{M}_1$.

**Proof of Theorem 1 Part 2**

Limit order traders take broker routing as given, following Lemmas 1 and 2. Following Lemma 3 and given routing to exchange $i$, a limit order trader will submit an order iff:

$$\theta_i (E[V|Ex_i] + y - (v - \Delta) - c) \geq 0$$ \hspace{1cm} (13)

Through substitution of $\theta_i$ from Proposition 1 and $E[V|Ex_i]$ from Proposition 2 and algebraic manipulation, the conditions for the markets 1 and 2 respectively are:
\[ y \geq c + \delta \cdot \sigma - \Delta \] (14)
\[ y \geq c + \frac{2\delta \cdot \sigma}{1 + \delta} - \Delta \] (15)

Denoting the conditions \( \bar{y}_1 \) and \( \bar{y}_2 \) at equality, any limit order trader with \( y \geq \bar{y}_i \) will satisfy this condition. These values exist for any parameter set following Assumptions 1 and 2.

**Proof of Proposition 1**

Since \( T_1 < T_2 \rightarrow M_1 > M_2 \), by lemma 1, brokers will route all small market orders to exchange 1, and all large market orders to both exchanges. Exchange 1 will receive orders any time a market order trader wishes to buy \( \frac{1}{2}(1 - \delta) \), while exchange 2 will receive orders any time a market order trader wishes to buy \( Q_2 = 2 \left( \frac{1}{4}(1 - \delta) \right) \). Both exchanges will receive an equal quantity of informed orders \( \frac{1}{2} \delta \). Thus, the execution probabilities at the two exchanges are:

\[
\theta_1 = \frac{1}{2} \delta + \frac{1}{2}(1 - \delta) \tag{16}
\]
\[
\theta_2 = \frac{1}{2} \delta + \frac{1}{4}(1 - \delta) \tag{17}
\]

Since \( 0 < \delta < 1 \), then \( \theta_1 > \theta_2 \).

**Proof of Proposition 2**

Consider a limit order buy. By an application of Bayes’ Rule, the expected value of the asset, conditional on execution at exchange \( i \) is:

\[
E[V|Ex_i] = \frac{Pr(V = \sigma, Ex_i) \cdot \sigma + Pr(V = 0, Ex_i) \cdot 0 + Pr(V = -\sigma, Ex_i) \cdot -\sigma}{Pr(V = \sigma, Ex_i) + Pr(V = 0, Ex_i) + Pr(V = -\sigma, Ex_i)} \tag{18}
\]
• With probability $\frac{1}{2}\delta$, $V = \sigma$. An informed trader arrives and picks off all orders at the ask. No orders at the bid execute, thus $Pr(V = \sigma, Ex_i) = 0$

• With probability $\frac{1}{2}(1 - \delta)$, $V = 0$, a liquidity trader arrives wishing to sell arrives. He always wishes to buy at least one unit, thus by Lemma 1, $Pr(V = 0, Ex_1) = \frac{1}{2}(1 - \delta)$.
With probability $\frac{1}{4}(1 - \delta)$ he wishes to buy two units, and $Pr(V = 0, Ex_2) = \frac{1}{4}(1 - \delta)$.

• With probability $\frac{1}{2}\delta$, $V = -\sigma$. An informed trader arrives and picks off all orders at the bid, thus $Pr(V = -\sigma, Ex_i) = \frac{1}{2}\delta$.

Through substitution of the above probabilities into Equation 18 and algebraic manipulation:

\[
E[V|Ex_1] = -\delta \cdot \sigma \quad (19)
\]
\[
E[V|Ex_2] = \frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} \quad (20)
\]

**Proof of Proposition 3**

See proof of Theorem 1, Part 2.

**Proof of Proposition 4**

Consider 2 near identical markets, such that $|T_1 - T_2|$ is small and $M_1 < \overline{M_1}$.

• If market 1 raises its maker fee sufficiently such that $M_1 > \overline{M_1}$:

  – Brokers will optimally route limit orders to market 2 via Lemma 2 and Theorem 1.

  – Less limit order traders will submit orders since $\overline{y}_1 < \overline{y}_2$

  – Each executed limit order will have a lower expected welfare gain since by Proposition 2 $E[V|Ex_1] > E[V|Ex_1]$.

  – Welfare will decline.
• If market 1 lowers its maker fee such that $M_1 < M_2$, market orders will preferentially be routed to exchange 2.

  – The $M_1 > \overline{M}_1$ becomes a condition for $M_2 > \overline{M}_2$ and the results from above are symmetric.

Lemma 4 (Increase in Adverse Selection Risk):

An increase in $\delta$:

1. Decreases the expected value of the asset for limit order buyers, given execution at all exchanges.
2. Lowers threshold $\overline{M}_1$.

Proof of Proposition 5

Follows from Lemma 4, proof of Theorem 1, Proposition 2 and $0 < \delta < 1, c > |M_i|$.

Proof of Proposition 6

As with Theorem 1, by Lemma 2, the broker will optimally route to exchange 1 iff:

$$\theta_1[c - M_1] \geq \theta_2[c - M_2]$$

(21)

By algebraic manipulation this gives the condition that:

$$c \geq \frac{2M_1 - (1 + \delta)M_2}{1 - \delta}$$

(22)

Denoting $c = \zeta$ when this condition holds with equality, the broker will optimally route any limit order to exchange 1 if $c \leq \zeta$.

Proof of Proposition 7

Follows from a comparison of the conditions in Propositions 2 and 6, as well as the welfare statement in Proposition 4.
B. Extension Equilibrium

Proof of Theorem 2 Part 1

Consider market makers at the bid. Given that brokers will first route market orders to exchange 1, it is optimal for the market makers to post a limit order at exchange 1 iff:

$$E[V|E_{x_1}] = -\delta \cdot \sigma - \Delta \geq M_1$$

Therefore $\tilde{M}_1$ is such that the above equation holds with equality.

If $M_1 \leq \tilde{M}_1$, a limit order will always be posted there, either by the market maker or a limit order trader. If this case, it is optimal for the market makers to post at exchange 2 iff:

$$E[V|E_{x_2}] = \frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} - \Delta \geq M_2$$

In this case, $\tilde{M}_2$ is such that the above equation holds with equality.

If $M_1 > \tilde{M}_1$, a limit order never be posted at exchange 1. In this case, limit order investors orders will always be routed to exchange 2 since $M_1 > M_2$ and $\theta_1(\Delta) = \theta_2(\Delta)$ if no market maker posts at exchange 1. If this case, it is optimal for the market makers to post at exchange 2 iff:

$$E[V|E_{x_2}] = -\delta \cdot \sigma - \Delta \geq M_2$$

In this case, $\tilde{M}_2$ is such that the above equation holds with equality.

Proof of Theorem 2 Part 2

Proof is similar to Theorem 1, Part 1.

The probabilities $\theta_1(\Delta), \theta_2(\Delta), \theta_1(2\Delta), \theta_2(2\Delta)$ are determined by the market maker behaviour established in Theorem 2, Section 1.

By Lemma 2, the broker will optimally route to exchange 1 iff:
\[ \theta_1(\Delta)[c - M_1] \geq \theta_2(\Delta)[c - M_2] \quad (26) \]

By algebraic manipulation this gives the condition that:

\[ M_1 \leq \frac{\theta_1(\Delta) - \theta_2(\Delta)}{\theta_1(\Delta)} c + \frac{\theta_2(\Delta)}{\theta_1(\Delta)} M_2 \quad (27) \]

Denoting \( M_1 = \overline{M}_1(\Delta) \) when this condition holds with equality, the broker will optimally route any limit order to exchange 1 if \( M_1 \leq \overline{M}_1(\Delta) \).

Proof for the price level \( 2\Delta \) follows through identical reasoning.

**Proof of Theorem 2 Part 3**

Proof is similar to Theorem 1, Part 2.

Limit order traders take broker routing as given, following Lemmas 1 and 2. Limit order traders will submit at price level \( \Delta \), knowing it will be routed to exchange \( i \), given:

\[ \theta_i(\Delta) \left( E[V|Ex_i] + y - (v - \Delta) - c \right) \geq \theta_j(2\Delta) \left( v + y - (v - 2\Delta) - c \right) \quad (28) \]

\[ \theta_i(\Delta) \left( E[V|Ex_i] + y - (v - \Delta) - c \right) \geq 0 \quad (29) \]

Which is to say that the expected value of submitting at \( \Delta \) is higher than that at \( 2\Delta \) and the value of abstaining. Since \( \theta_1(\Delta), \theta_2(\Delta), \theta_1(2\Delta), \theta_2(2\Delta), E[V|Ex_1], E[V|Ex_2] \) and broker routing decisions are already determined by market maker behaviour, the condition \( \overline{y}_i(\Delta) \) can be determined by evaluating the two above conditions at equality, and selection whichever is more stringent.

\[ \overline{y}_1(\Delta) = \max \left\{ \frac{\theta_j(2\Delta)(2\Delta - c) - \theta_i(\Delta)(E[V|Ex_i] - (v - \Delta) - c)}{\theta_i(\Delta) - \theta_j(2\Delta)}, -E[V|Ex_i] + (v - \Delta) + c \right\} \quad (30) \]

Algebraic manipulation can show that:
\[
\frac{\theta_j(2\Delta)(2\Delta - c) - \theta_i(\Delta)(E[V|Ex_i] - (v - \Delta) - c)}{\theta_i(\Delta) - \theta_j(2\Delta)} \geq -E[V|Ex_i] + (v - \Delta) + c
\] 

(31)

and thus:

\[
\overline{y}_1(\Delta) = \frac{\theta_j(2\Delta)(2\Delta - c) - \theta_i(\Delta)(E[V|Ex_i] - (v - \Delta) - c)}{\theta_i(\Delta) - \theta_j(2\Delta)}
\]

(32)

For traders with \( y < \overline{y}_1 \), they will optimally submit at price level \( 2\Delta \) if:

\[
\theta_i(2\Delta)(v + y - (v - 2\Delta) - c) \geq 0
\]

(33)

The condition \( \overline{y}_i(2\Delta) \) can be determined by evaluating the preceding condition at equality, in order to obtain:

\[
\overline{y}_i(2\Delta) = -2\Delta + c
\]

(34)

**Proof of Proposition 8 Part 1**

Consider an increase from \( M_1 \leq \tilde{M}_1 \) to \( M_1' > \tilde{M}_1 \). This violates the condition \( E[V|Ex_1] = -\delta \cdot \sigma - \Delta \geq M_1 \) and it is no longer optimal for market makers to post.

Since market makers are no longer posting at exchange 1, the expected value at exchange 2 changes such that \( E[V|Ex_2] = -\delta \cdot \sigma \). Since \( M_1 > M_2 \), then \( E[V|Ex_2] = -\delta \cdot \sigma - \Delta \geq M_2 \).

It is now optimal for market makers to post at exchange 2.

**Proof of Proposition 8 Part 2**

Consider an increase from \( M_2 \leq \tilde{M}_2 \) to \( M_2' > \tilde{M}_2 \). This violates either the condition \( E[V|Ex_2] = \frac{-\delta \cdot \sigma}{\delta + 0.5 \cdot (1 - \delta)} - \Delta \geq M_2 \) or \( E[V|Ex_2] = -\delta \cdot \sigma - \Delta \geq M_2 \), depending on whether market makers are posting at exchange 1. It is no longer optimal for market makers to post at exchange 2.

**Proof of Proposition 9 Part 1** Consider \( \overline{M}_1 < M_1 \leq \tilde{M}_1 \). Brokers are routing limit
orders to exchange 2. If the fees at exchange 1 increase to $M'_1 > \tilde{M}_1$, brokers will continue to route to exchange 2, however $\theta_2(\Delta)$ increases as there are no longer any limit orders from market makers at exchange 1. The value $E[V|Ex_2]$ also improves for limit order investors, as a higher concentration of uninformed market orders ($\frac{1}{2}(1-\delta)$ as opposed to $\frac{1}{4}(1-\delta)$) reach exchange 2. Therefore, all limit order traders posting at exchange 2 have higher utility in expectation.

In addition, $\theta_1(2\Delta)$ increases from 0 to $\frac{1}{4}(1-\delta)$. Some measure of traders, between $\overline{y}_i(\Delta)$ and $\overline{y}_i(2\Delta)$ will begin submitting orders at the price level $2\Delta$. Each of these traders will also be better off in expectation.

**Proof of Proposition 9 Part 2**

Consider $M_2 < M_1 \leq \tilde{M}_2$, increasing to $M'_2 > \tilde{M}_2$. If $M_1 \leq \tilde{M}_1$, there is still liquidity available at the best. $\theta_1(\Delta)$, $\theta_2(\Delta)$, $E[V|Ex_1]$ and $E[V|Ex_2]$ remain the same, however $\theta_1(2\Delta)$ increases from 0 to $\frac{1}{4}(1-\delta)$. Some measure of traders, between $\overline{y}_i(\Delta)$ and $\overline{y}_i(2\Delta)$ will begin submitting orders at the price level $2\Delta$. Each of these traders will also be better off in expectation.

If $M_1 > \tilde{M}_1$, there is now no liquidity available at the best. $\theta_1(2\Delta)$ increases from $\frac{1}{4}(1-\delta)$ to $\frac{1}{2}(1-\delta)$ and $\theta_2(2\Delta)$ increases from 0 to $\frac{1}{4}(1-\delta)$. Each trader submitting an order at price level $2\Delta$ is better off.

**VII. Appendix 2: Extension Simulations**

**A. Simulation Graphical Parameters**

All graphs are based on the following parameters:

- $v = 0$
- $\Delta = 1$
- $\delta = 0.5$
- $\sigma = 1.5$
\[ b = 0.5 \]
\[ y = 2 \]